

# A weakly nonlinear analysis of the magnetorotational instability in a model channel flow

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The magnetorotational instability (MRI) is generally accepted as the physical cause of turbulence and enhanced transport of angular momentum in accretion disks, which are one of the most basic and ubiquitous astrophysical structures. We show, by means of a perturbative weakly nonlinear analysis, that the axisymmetric MRI of a viscous, resistive, incompressible rotating shear flow subject to a background vertical magnetic field in a thin channel gives rise to a Landau equation for the disturbance amplitude. For small magnetic Prandtl number ( $\mathcal{P}_m$ ), the saturation amplitude is  $\propto \sqrt{\mathcal{P}_m}$  and the resulting momentum transport scales as  $\mathcal{R}^{-1}$ , where  $\mathcal{R}$  is the *hydrodynamic* Reynolds number. Simplifying assumptions, such as linear shear base flow, mathematically expedient boundary conditions and continuous spectrum of the vertical linear modes, are used to facilitate this analysis. The asymptotic results are shown to comply with numerical calculations using a spectral code. They suggest that the transport due to the nonlinearly developed MRI may be very small in experimental setups with  $\mathcal{P}_m \ll 1$ .

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The MRI has acquired growing theoretical and experimental interest in recent years, moving beyond just the astrophysical community. The linear MRI has been known for almost 50 years [1, 2]: Rayleigh stable rotating Couette flows of conducting fluids are destabilized in the presence of a vertical magnetic field if  $d\Omega^2/dr < 0$  (angular velocity decreasing outward). However, the MRI only acquired importance to astrophysics after the influential work of Balbus & Hawley [3], who demonstrated its viability to accretion disks (for any sufficiently weak field) with linear analysis supplemented by nonlinear numerical simulations. Enhanced transport (conceivably turbulent) is necessary for accretion to proceed in disks found in a variety of astrophysical settings - from protostars to active galactic nuclei. The difficulty in identifying sources of such enhanced transport has been a limitation to proper theoretical understanding and modeling of these objects. Thus the MRI has been widely accepted as an attractive solution for enhanced transport (see the comprehensive reviews [4, 5]) even though some questions on the nature of MRI-driven turbulence remain [6]. Because of the MRI's importance and some of its outstanding unresolved issues, several groups have recently undertaken projects to investigate the instability under laboratory conditions [7, 8, 9]. Additionally, numerical simulations specially designed for experimental setups have been conducted.

In this Letter we perform a weakly nonlinear analysis of the MRI near threshold for a rotating flow in channel geometry, subject to idealized but mathematically expedient boundary conditions. This kind of approach is important because the viability of this linear instability as the driver of turbulence and angular-momentum trans-

port relies on understanding its nonlinear development and saturation. By complementing the above mentioned simulations, analytical methods remain useful to gain further physical insight. To facilitate an analytical approach we make a number of simplifying assumptions so as to make the system amenable to well-known methods [10, 11, 12, 13] for the derivation of nonlinear envelope equations (i.e. with an amplitude weakly dependent on time *and* space). Additionally we assume a narrow channel geometry, i.e. the small-gap limit of Taylor-Couette flow.

The hydromagnetic equations in cylindrical coordinates [14] are applied to the neighborhood of a representative point in the system, using the shearing box (SB) approximation [15]. The SB has been employed in numerous (analytical and numerical) studies of the MRI in accretion disks and are also appropriate for this model problem. Within the framework of the *small* shearing box (SSB)[16], the base flow is steady and incompressible with constant pressure. The velocity  $\mathbf{V} = U(x)\hat{\mathbf{y}}$  has a linear shear profile  $U(x) = -qU_0x$ , representing an axisymmetric flow about a point  $r_0$ , that rotates with a rate  $\Omega_0$  defined from the differential rotation law  $\Omega(r) \propto \Omega_0(r/r_0)^{-q}$ . We restrict ourselves to a constant initial magnetic field  $\mathbf{B} = B_0\hat{\mathbf{z}}$ . Cartesian coordinates  $x, y, z$  are used to represent the radial (shear-wise), azimuthal (stream-wise) and vertical directions, respectively. The base flow is axisymmetrically disturbed by perturbations of the velocity,  $\mathbf{u} = (u_x, u_y, u_z)$ , magnetic field,  $\mathbf{b} = (b_x, b_y, b_z)$ , and total pressure,  $\varpi$ . This results, after non-dimensionalization, in the following set

of *non-linear* equations:

$$\frac{d\mathbf{u}}{dt} - 2\hat{\mathbf{z}} \times \mathbf{u} - q\mathbf{u}_x \hat{\mathbf{y}} - \mathcal{C} \hat{\mathbb{B}} \mathbf{b} = -\nabla \varpi + \frac{1}{\mathcal{R}} \nabla^2 \mathbf{u} \quad (1a)$$

$$\frac{d\mathbf{b}}{dt} - \hat{\mathbb{B}} \mathbf{u} + q\mathbf{b}_x \hat{\mathbf{y}} = \frac{1}{\mathcal{R}_m} \nabla^2 \mathbf{b} \quad (1b)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{b} = 0, \quad (1c)$$

where  $\hat{\mathbb{B}} \equiv (\mathbf{b} \cdot \nabla + B_0 \partial_z)$ . Lengths were scaled by  $\tilde{L}$  (the channel width scale), time by  $\tilde{\Omega}_0^{-1} \equiv \Omega_0^{-1}$  and the magnetic field by  $\tilde{B}_0 \equiv B_0$ . The quantities marked by tilde are thus dimensional. With this scaling,  $B_0 = 1$  in (1c) but we retain it for later convenience. The non-dimensional parameter  $\mathcal{C} \equiv \tilde{B}_0^2 / (4\pi \tilde{\rho}_0 \tilde{\Omega}_0^2 \tilde{L}^2) \equiv \tilde{V}_A^2 / \tilde{U}^2$ , is the *Cowling number*, measuring the dynamical importance of the magnetic field. In addition the *Reynolds number*,  $\mathcal{R} \equiv \tilde{\Omega}_0 \tilde{L}^2 / \tilde{\nu}$ , and the *magnetic Reynolds number*,  $\mathcal{R}_m \equiv \tilde{\Omega}_0 \tilde{L}^2 / \tilde{\eta}$ , are used, where  $\tilde{\nu}$  and  $\tilde{\eta}$  are the microscopic kinematic viscosity and magnetic resistivity of the fluid, respectively. We introduce two additional non-dimensional parameters, which figure in what follows: the *Elsasser number*,  $\mathcal{S} \equiv \mathcal{R}_m \mathcal{C} = \tilde{B}_0^2 / (4\pi \tilde{\rho}_0 \tilde{\Omega}_0 \tilde{\eta})$ , and the *magnetic Prandtl number*,  $\mathcal{P}_m \equiv \mathcal{R}_m / \mathcal{R}$ .

Linearization of (1), for perturbations of the form  $\propto e^{st+ik_x x+ik_z z}$ , gives rise to the dispersion relation,

$$\mathcal{D}(s; k_x, k_z, \mathcal{P}_m, \mathcal{S}, \mathcal{C}, q) = a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0,$$

where all the  $a_i$  are functions of  $k_x$ ,  $k_z$  and the other parameters, but we shall write explicitly only the coefficient that will be used in what follows,

$$a_4 = \frac{\mathcal{C}}{\mathcal{S}^4} \left[ k_T^2 \mathcal{C} (\mathcal{C} k_T^4 \mathcal{P}_m + k_z^2 \mathcal{S}^2)^2 + \kappa^2 \mathcal{S}^2 \mathcal{C} k_T^4 k_z^2 - 2q \mathcal{S}^4 k_z^4 \right], \quad (2)$$

where the notation  $\kappa^2 \equiv 2(2-q)$ ,  $k_T^2 \equiv k_x^2 + k_z^2$  is introduced.

For given values of the parameters (we denote them collectively as  $\Pi$ ) there will be four distinct modes. The linear theory in various limits for this problem has been discussed in numerous publications (see [4, 5]) and we shall not elaborate upon them any further. Rather, we focus on situations where the most unstable mode (of the four) is marginal (critical) for some  $k_z$  at a given value of  $k_x = K$ . We fix  $K$  in order to focus on the marginal vertical dynamics within a channel (see below).

Marginality to the MRI ( $s = 0$ ) implies the vanishing of the real coefficient  $a_4(k_z)$  (as expressed above) and its derivative with respect to  $k_z$  at some  $k_z = Q$ :

$$a_4(k_z = Q; K, \Pi) = 0 \quad \text{and} \quad \frac{\partial a_4}{\partial k_z}(k_z = Q; K, \Pi) = 0. \quad (3)$$

The second condition and (2) yield

$$\begin{aligned} & Q \left[ Q^2 \mathcal{S}^4 (\mathcal{C} Q^2 - 4q) + (K^2 + Q^2) 2Q^2 \mathcal{C} \mathcal{S}^2 (\mathcal{S}^2 + \kappa^2) \right. \\ & \left. + (K^2 + Q^2)^2 \mathcal{C} \mathcal{S}^2 (6\mathcal{C} \mathcal{P}_m Q^2 + \kappa^2) + 2(K^2 + Q^2)^3 \mathcal{P}_m \mathcal{C}^2 \mathcal{S}^2 \right. \\ & \left. + 5(K^2 + Q^2)^4 \mathcal{P}_m^2 \mathcal{C}^3 \right] = 0. \end{aligned} \quad (4)$$

This equation, together with  $a_4(k_z = Q, K; \Pi) = 0$ , can be solved for  $\mathcal{S}$  and  $Q$ . The general expressions for  $\mathcal{S}(K, \mathcal{P}_m, \mathcal{C}, q)$  and  $Q(K, \mathcal{P}_m, \mathcal{C}, q)$  are lengthy but their asymptotic forms, for  $\mathcal{P}_m \ll 1$ , are simple,

$$\mathcal{S} = \frac{\sqrt{16 \mathcal{C} q (2-q)} K}{(2q - \mathcal{C} K^2)} + \mathcal{O}(\mathcal{P}_m), \quad Q^2 = K^2 \frac{2q - \mathcal{C} K^2}{2q + \mathcal{C} K^2} + \mathcal{O}(\mathcal{P}_m). \quad (5)$$

If  $\mathcal{C} K^2 > 2q$  the solutions are not physically meaningful, while the case  $\mathcal{C} K^2 = 2q$  corresponds to the ideal MRI limit.

We consider the properties of the MRI within the confines of an idealized model channel geometry with walls located at  $x = 0, \pi/K$ . We allow all quantities to be vertically periodic on a scale  $L_z$  commensurate with integer multiples of  $2\pi/Q$ . As long as  $L_z \gg 1/Q$ , the limit of a vertically extended system (and thus a continuous spectrum of vertical modes) may be effected.

We follow the fluid into instability by tuning the vertical background field away from the steady state, i.e.  $B_0 \rightarrow 1 - \epsilon^2 \tilde{\lambda}$ , where  $\epsilon \ll 1$  and  $\tilde{\lambda}$  is an  $\mathcal{O}(1)$  control parameter. This means that we are now in a position to apply procedures of singular perturbation theory to this problem, by employing the multiple-scale (in  $z$  and  $t$ ) method (e.g., [10]). It facilitates, by imposing suitable solvability conditions at each expansion order of the calculation in order to prevent a breakdown in the solutions, a derivation of an envelope equation for the unstable mode. Thus for any dependent fluid quantity  $F(x, z, t)$  we assume

$$F(x, z, t) = \epsilon F_1(x, z, t) + \epsilon^2 F_2(x, z, t) + \epsilon^3 F_3(x, z, t) + \dots \quad (6)$$

The fact that the  $x$  and  $z$  components of the velocity and magnetic field perturbations can be derived from a streamfunction,  $\Psi(x, z)$  and magnetic flux function  $\Phi(x, z)$  reduces the number of relevant dependent variables  $F$  to four ( $u_y$  and  $b_y$  are the additional two). These four dependent variables will also be used in the spectral numerical calculation (see below).

For the lowest  $\epsilon$  order of the equations, resulting from substituting the expansions into the original PDE and collecting same order terms, we make the Ansatz  $F_1(x, z, t) = \hat{F}_1 \tilde{A}(\epsilon z, \epsilon^2 t) e^{iQz} \sin Kx + \text{c.c.}$ , where  $\hat{F}_1$  is a constant (according to the variable in question), and where the envelope function  $\tilde{A}$  (an arbitrary constant amplitude in linear theory) is now allowed to have weak space (on scale  $Z \equiv \epsilon z$ ) and time (on scale  $T \equiv \epsilon^2 t$ ) dependencies. Because this system is tenth order in  $x$ -derivatives, a sufficient number of conditions must be specified at the edges. The  $\sin Kx$  functional dependence of the Ansatz satisfies the ten boundary conditions

$$u_x = u_y = b_x = b_y = \partial_x u_z = 0, \quad (7)$$

at  $x = 0, \pi/K$ . These conditions are also subsequently satisfied to all orders of the expansion procedure because

of the symmetry property of  $\sin Kx$  under the nonlinear operations of (1). These specifications are a mixture of kinematic conditions on the edges: no-slip conditions on  $u_x$  and  $u_y$  with free-slip conditions on  $u_z$ . The conditions are similarly mixed for the magnetic field: conducting conditions on  $b_x$  and insulating conditions on  $b_y$ . Although these conditions are idealized, they are mathematically expedient, in that the coefficients of the resulting Landau equation may be expressed analytically in the limit  $\mathcal{P}_m \ll 1$ , and they serve to capture the salient features of the resulting dynamics (see below).

The end result of the asymptotic procedure procedure is the well-known Landau Equation (LE) which, for  $\mathcal{P}_m \ll 1$ , is

$$\partial_T A = \lambda A - \frac{1}{\mathcal{P}_m \mathcal{C}} A^3 + D \partial_z^2 A. \quad (8)$$

Here  $A \equiv \sqrt{\xi} \tilde{A}$ ,  $\lambda \equiv \zeta \tilde{\lambda}$ , and the coefficients, for a value of  $q = 3/2$  (corresponding to a Keplerian shear flow) are

$$\xi = \frac{3}{4} \cdot \frac{5\mathcal{S}^4 - 18\mathcal{S}^2 - 32 + 2(\mathcal{S}^2 + 16)\sqrt{\mathcal{S}^2 + 1}}{\mathcal{S}(\mathcal{S}^2 + 1)(4\sqrt{\mathcal{S}^2 + 1} - 3)}, \quad (9a)$$

$$D = 6 \frac{(\mathcal{S}^2 + 2 - 2\sqrt{\mathcal{S}^2 + 1})(\mathcal{S}^2 + 1)}{\mathcal{S}^3(4\sqrt{\mathcal{S}^2 + 1} - 3)}, \quad (9b)$$

$$\zeta = \frac{3\mathcal{S}\sqrt{\mathcal{S}^2 + 1} - 6\mathcal{S}}{4\mathcal{S}^2 + 1 + \sqrt{\mathcal{S}^2 + 1}}. \quad (9c)$$

The expressions for a general  $q$  are very long and involved. For  $\mathcal{S} \gg 1$ , (i.e. as one approaches the ideal MRI limit), these simplify to  $\xi = 15/16$ ,  $\zeta = 3/4$ ,  $D = 3/2$ ; and in general they remain  $\mathcal{O}(1)$  quantities for all reasonable values of  $\mathcal{S}$ .

Before proceeding any further, it is important to note that equations (1) admit an integral relationship for the total disturbance energy. Application of the boundary conditions (7) yields a hydromagnetic extension of the Reynolds-Orr equation [18]. Here it takes on the form:

$$\begin{aligned} \frac{dE_V}{dt} = & -\frac{dU}{dx} \int (u_x u_y - C b_x b_y) dx dz \\ & - \int \left( \frac{1}{\mathcal{R}} |\nabla \mathbf{u}|^2 + \frac{1}{\mathcal{R}_m} \mathcal{C} |\nabla \mathbf{b}|^2 \right) dx dz, \end{aligned} \quad (10)$$

where  $|\nabla \mathbf{u}|^2 \equiv \sum_\mu |\nabla u_\mu|^2$  for  $\mu = x, y, z$  and similarly for  $\mathbf{b}$ .  $E_V$  is the total energy, per unit length in the azimuthal direction, of the disturbances in the domain,  $E_V \equiv \frac{1}{2} \int (\mathbf{u}^2 + C \mathbf{b}^2) dx dz$ , and the first integral (multiplied by  $dU/dx$ ) represents the energy fed into the hydrodynamical and magnetic disturbances by the background shear. The integrand  $T \equiv u_x u_y - C b_x b_y$  is the relevant component of the total stress in the disturbance. Integrating it along the wall and averaging radially gives an estimate of the average angular momentum transport rate inside the channel, i.e.  $\dot{J} \equiv (K/\pi) \int_{-L_z/2}^{L_z/2} \int_0^{\pi/K} T dx dz$ . A positive measure for  $\dot{J}$  means to say that the transport is outwards.

We can use our asymptotic solutions to express  $E_V$  and  $\dot{J}$  in terms of the envelope function  $A(Z, T)$ . To the first few leading orders in  $\epsilon$  (recalling that  $\mathcal{P}_m \ll 1$ ) we find that

$$E_V = \epsilon^2 \alpha(\mathcal{S}) \mathcal{C}^2 A^2 + \epsilon^4 \mathcal{C}^3 \left[ \frac{\beta_2(\mathcal{S})}{\mathcal{P}_m^2} + \frac{\beta_1(\mathcal{S})}{\mathcal{P}_m} \right] A^4 + \dots \quad (11)$$

$\alpha$  and  $\beta_i$  are functions whose value is of  $\mathcal{O}(1)$  for all reasonable values of  $\mathcal{S}(K, q = 3/2)$ . The two bracketed terms arise from the second order (in  $\epsilon$ ) azimuthal velocity disturbances. Contributions to the total angular momentum transport ( $\dot{J} = \dot{J}_H + \dot{J}_B$ ) due to the hydrodynamic ( $\dot{J}_H$ ) and magnetic correlations ( $\dot{J}_B$ ) are, to leading order,

$$\dot{J}_H = \frac{9\epsilon^2}{\mathcal{S}} \left( \frac{2 + \mathcal{S}^2 - 2\sqrt{1 + \mathcal{S}^2}}{1 + \mathcal{S}^2} \right) A^2 + \mathcal{O}(\epsilon^3, \epsilon^2 \mathcal{P}_m) \quad (12a)$$

$$\dot{J}_B = \frac{3\epsilon^2}{\mathcal{S}} \left( 1 - \frac{1}{\sqrt{1 + \mathcal{S}^2}} \right) A^2 + \mathcal{O}(\epsilon^3, \epsilon^2 \mathcal{P}_m). \quad (12b)$$

The envelope  $A$  is found by solving the LE, which is a well-studied system (see, e.g., [13] for a summary and references). It has three steady uniform solutions in one spatial dimension (here, the vertical): an unstable state  $A = 0$ , and two stable states  $A = \pm A_s$ , where  $A_s^2 = |\zeta(\mathcal{S})| \mathcal{P}_m \mathcal{C}$  (note that  $\zeta(\mathcal{S})$  is an  $\mathcal{O}(1)$  quantity).  $A_s$  is also the saturation amplitude, because the system will develop towards it [13]. Setting  $A \rightarrow A_s$  in (12a-b) and in the expression for  $E_V$ , followed by some algebra, reveals that the angular momentum flux in the saturated state is, to leading order in  $\mathcal{P}_m$  and  $\epsilon$ ,  $\dot{J} = \epsilon^2 |\zeta(\mathcal{S})| \gamma(\mathcal{S}) \mathcal{P}_m \mathcal{C} \mathcal{S}^{-1}$ , while in the expression for the energy one term is independent of  $\mathcal{P}_m$ ,  $E_V = \epsilon^4 \mathcal{C}^3 \zeta^2(\mathcal{S}) \beta_2(\mathcal{S}) + \mathcal{O}(\mathcal{P}_m)$ , where  $\gamma(\mathcal{S}) = 12 - 21(1 + \mathcal{S}^2)^{-1/2} + 9(1 + \mathcal{S}^2)^{-1}$ .

The key results of this idealized analysis are thus,

$$A_s \sim \sqrt{\mathcal{P}_m \mathcal{C}} \longrightarrow E_V \sim E_0, \quad \dot{J} \sim \mathcal{R}^{-1}, \quad (13)$$

for  $\mathcal{P}_m \ll 1$ , where  $E_0$  is a constant, independent of  $\mathcal{P}_m$ .

We have also performed numerical calculations, using a 2-D spectral code to solve the original nonlinear equations (1), in the streamfunction and magnetic flux function formulation, near MRI threshold. The asymptotic theory reflects the trends seen in these simulations. The code implements a Fourier-Galerkin expansion in  $64 \times 64$  modes in each of the four independent physical variables, i.e.  $\mathbf{F} = \sum_{n,m} \mathbf{F}_{nm}(t) \sin K_n x e^{iQ_m z} + \text{c.c.}$ , where  $\mathbf{F} = (\psi, u_y, \Phi, b_y)^T$  and where  $\mathbf{F}_{nm}(t)$  is the time-dependent amplitude of the particular Fourier-Galerkin mode in question (denoted by the indices  $n, m$ ).

We typically start with white noise initial conditions on  $\mathbf{F}_{nm}(0)$  at a level of 0.1 the energy of the background shear. Because of space limitations we display only a representative plot of runs made with  $\mathcal{S} = 5.0$ ,  $\mathcal{C} = 0.05$ , (i.e.  $\mathcal{R}_m$  fixed),  $q = 3/2$  and  $\epsilon^2 = 0.2$  for a few successively increasing values of  $\mathcal{R}$  (see Figure 1). Note that

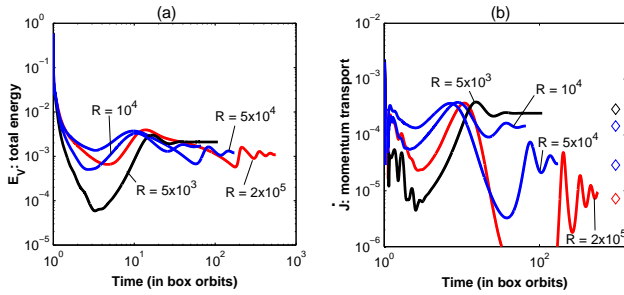


FIG. 1:  $E_v$  (panel a) and  $\dot{J}$  (panel b) as a function of time from numerical calculation for  $\mathcal{R} = 5 \times 10^3, 10^4, 5 \times 10^4, 2 \times 10^5$ . The diamonds in panel (b) show the scaling predicted by our asymptotic analysis, which also predicts a constant final value of the disturbance energy, as apparent in panel (a). Given that  $\dot{J} > 0$ , it means that the angular momentum transport is outwards (positive x-direction). One box orbit =  $2\pi$  in nondimensionalized units.

with this value of  $\epsilon$  the fastest growing linear mode has a growth rate of  $\sim 0.065$  in these units, and thus fully developed ideal MRI can not be expected. It is apparent that  $E_v$  saturates at a constant independent of  $\mathcal{R}$  (for large enough  $\mathcal{R}$ , i.e. small enough  $\mathcal{P}_m$ ), while  $\dot{J}$ , showing outward transport, saturates at values that scale as  $\mathcal{R}^{-1}$ , as expected from our asymptotic analysis.

The numerical and asymptotic solutions developed here show that in the saturated state the second order azimuthal velocity perturbation becomes dominant over all other quantities for  $\mathcal{P}_m \ll 1$ : it has the spatial form  $\sin 2Kx$ , independent of  $z$ . It arises from the  $\mathcal{O}(\epsilon^2)$  term in asymptotic expansion (6), and appears to be the primary agent in the nonlinear saturation of the MRI in the channel. It acts anisotropically so as to modify the shear profile and results in a non-diagonal stress component (relevant for radial angular momentum transport), which behaves like  $\sim 1/\mathcal{R}$ .

These results should be compared to experiments and their numerical simulations. Thus far no experimental results for this geometry have been reported but corresponding numerical simulations do exist [20]. Our results on the scaling of transport (decreasing with increasing  $\mathcal{R}$  and independent of  $\mathcal{R}_m$  for  $\mathcal{P}_m \ll 1$ ) are qualitatively consistent with these numerical simulations but more careful comparisons are needed. In addition, issues of the effect of numerical resolution upon the resulting dynamics must be considered, as is done in the turbulent dynamo problem [22, 23], for example. Our analysis is complementary to that of Knobloch and Julien [21] who have recently reported the results of an asymptotic MRI analysis for a developed state far from marginality.

The trends predicted by this simplified model are not qualitatively altered by the appearance of boundary layers which arise when no-slip, perfectly conducting boundary conditions are enforced in the limit  $\mathcal{P}_m \ll 1$ . Calcula-

tions with these more realistic boundary conditions lead to amplitude equations, whose coefficients can not be, however, expressed analytically in any convenient way. A full presentation of such a calculation will appear in a forthcoming work.

Further analytical investigations of the nonlinear MRI will contribute toward assembling a hierarchical understanding of this important instability, from laboratory experiments to numerical simulations to astrophysical disks.

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